INTERPOLATING BASIS IN THE SPACE $C^\infty[-1,1]^d$

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Abstract. An interpolating Schauder basis in the space $C^\infty[-1,1]^d$ is suggested. In the construction we use Newton's interpolation of functions at the sequence that was found recently by Jean-Paul Calvi and Phung Van Manh.

1. **Introduction**

There is a variety of different topological bases in the space $C^\infty[-1,1]$. The classical work here is [14], where Mityagin found the first such basis, namely the Chebyshev polynomials. Later it was proven in [12] and [1] that other classical orthogonal polynomials have the basis property in this space as well. If we apply the result by Zeriahi [18] to the set $[-1,1]$ then we obtain a basis from polynomials that are orthogonal with respect to more general measures. Following Triebel [17] (see also [2]) a basis from eigenvectors of a certain differential operator can be constructed in this space. A special basis in $C^\infty[0,1]$ was used in [7] to construct a basis in the space of $C^\infty-$ functions on a graduated sharp cusp with arbitrary sharpness. Recently it was shown in [9] that the wavelets system suggested by T.Kilgor and J.Prestin in [13] also forms a basis in the space $C^\infty[-1,1]$.

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In view of the isomorphism

$$C^\infty[-1,1]^d \simeq \underbrace{C^\infty[-1,1]\,\hat{\otimes}\,C^\infty[-1,1]\,\hat{\otimes}\,\cdots\,\hat{\otimes}\,C^\infty[-1,1]}_{d}$$

([11], Ch.2, T.13) these results can be extended to the multivariate case.

Here we present an interpolating topological basis in the space $C^\infty[-1,1]^d$. Together with [8] it gives a unified approach for constructing bases in spaces of infinitely differentiable functions and their traces on compact sets.

A polynomial basis $(P_n)_{n=0}^\infty$ in a functional space is called a Faber (or strict polynomial) basis if $deg\, P_n = n$ for all n. Due to the classical result of Faber [6], the space $C[a,b]$ does not possess such a basis.

Here we use the Newton interpolation, so the basis presented for $C^\infty[-1,1]$ is a Faber basis. The crucial aspect in the proof is the existence of the sequence $(x_n)_{n=1}^\infty \subset [-1,1]$ with a moderate growth of the corresponding Lebesgue constants. A sequence of this type was found recently by Jean-Paul Calvi and Phung Van Manh in [4]. This essentially improves the author's result [10] where, for the sequence of the Lebesgue constants, only the asymptotic behavior $\exp(\log^2 n)$ was achieved.

2. Interpolating topological basis in $C^\infty[-1,1]^d$

Let $\mathcal{X}$ be a linear topological space over the field $\mathbb{K}$. By $\mathcal{X}'$ we denote the topological dual space. A sequence $(e_n)_{n=0}^\infty \subset \mathcal{X}$ is a (topological) basis for $\mathcal{X}$ if for each $f \in \mathcal{X}$ there is a unique sequence $(\xi_n(f))_{n=0}^\infty \subset \mathbb{K}$ such that the series $\sum_{n=0}^\infty \xi_n(f)\, e_n$ converges to f in the topology of $\mathcal{X}$. The sequence $(\xi_n)_{n=0}^\infty$ of linear functionals $\xi_n : \mathcal{X} \longrightarrow \mathbb{K} : f \mapsto \xi_n(f)$ for $n \in \mathbb{N}_0 := \{0, 1, \cdots\}$ is biorthogonal to $(e_n)_{n=0}^\infty$ and total over $\mathcal{X}$. The latter indicates that $\xi_n(f) = 0$ for all $n \in \mathbb{N}_0$ implies $f = 0$.

Given a compact set $K \subset \mathbb{R}$ and a sequence of distinct points $(x_n)_1^\infty \subset K$, let $e_0 \equiv 1$ and $e_n(x) = \Pi_1^n(x - x_k)$ for $n \in \mathbb{N}$. Let $\mathcal{X}(K)$ be a Fréchet space of continuous functions on K, containing all polynomials. By ξ_n we denote, by means of the divided differences, the linear functional $\xi_n(f) = [x_1, x_2, \cdots, x_{n+1}]f$ with $f \in \mathcal{X}(K)$ and $n \in \mathbb{N}_0$. Properties of the divided differences (see e.g. [5]) imply the following evident result.

Lemma 1. *If a sequence $(x_n)_1^\infty$ of distinct points is dense on a perfect compact set $K \subset \mathbb{R}$, then the system $(e_n, \xi_n)_{n=0}^\infty$ is biorthogonal and the sequence of functionals $(\xi_n)_{n=0}^\infty$ is total on $\mathcal{X}(K)$.*

Here the partial sum of the expansion with respect to the system $(e_n, \xi_n)_{n=0}^\infty$ is the Lagrange interpolating polynomial of f, so $L_n(f,x) = \sum_{k=0}^n \xi_k(f)\ e_k$, and $L_n : \mathcal{X}(K) \to \mathcal{P}_n : f \mapsto L_n(f, \cdot)$ is the corresponding projection on the space of all polynomials of degree at most n.

We proceed to present an interpolating basis in the space $\mathcal{X} = C^\infty[-1,1]$ equipped with the topology defined by the sequence of norms

$$\| f \|_p = sup\,\{|f^{(i)}(x)| : |\,x\,| \le 1,\ 0 \le i \le p\}, \qquad p \in \mathbb{N}_0.$$

Let $(x_n)_1^\infty$ be the sequence in $[-1,1]$ suggested in [4]. Then, by Th.3.1 in [4], the sequence of uniform norms of L_n, which are the Lebesgue constants corresponding to the sequence $(x_n)_1^\infty$, is polynomially bounded:

$$||L_n||_0 \le C\, n^3\, \log n \tag{1}$$

for some constant C.

Theorem 1. *The functions $(e_n)_{n=0}^\infty$ form a topological basis in the space $C^\infty[-1,1]$.*

Proof:

Since the space under consideration is complete, it is enough to show that, given $f \in C^\infty[-1,1]$, the series $\sum_{n=0}^\infty \xi_n(f)\, e_n$ absolutely converges, that is the series $\sum_{n=0}^\infty |\, \xi_n(f)\,| \cdot \|\, e_n\, \|_p$ converges for each $p \in \mathbb{N}$.

By the Markov inequality (see e.g. [5], p.98),

$$|\, \xi_n(f)\,| \cdot \|\, e_n\, \|_p = ||L_n(f) - L_{n-1}(f)||_p \le n^{2p}\, ||L_n(f) - L_{n-1}(f)||_0. \tag{2}$$

Let Q_n be the polynomial of best uniform approximation to f on $[-1,1]$ and $E_n(f) = \|\, f - Q_n\|_0$. By the Jackson theorem (see e.g. [5], p.219), the sequence $(E_n(f))_{n=0}^\infty$ is rapidly decreasing, that is $n^q\, E_n(f) \to 0$, as $n \to \infty$ for any fixed q. Thus, for each q there is a constant C_q such that $E_n(f) \le C_q\, n^{-q}$ for all $n \in \mathbb{N}$.

Applying (1) and a standard argument we have $||L_n(f) - f||_0 \le ||L_n(f) - L_n(Q_n)||_0 + ||Q_n - f||_0 \le (C\, n^3 \log n + 1)\, C_q\, n^{-q}$. Therefore, $||L_n(f) - L_{n-1}(f)||_0 \le (C\, n^3 \log n + 1)\,(C_q + C_{q-1})\,(n-1)^{-q}$. From (2) we conclude that the value $q = 2p + 5$ provides the desired result. □

Corollary 1. *The space $C^\infty[-1,1]^d$ possesses an interpolating topological basis.*

Indeed, by Th.16 in [3], there is a sequence of points in $[-1,1]^d$ for a multivariate Newton interpolation with a polynomial grows of the corresponding Lebesgue constants. The tensor products of ordinary divided differences work now as biorthogonal functionals. Since for the set $[-1,1]^d$ both Markov's type estimation and Jackson's theorem are valid (see e.g. [15] and [16] 5.3.2), we can repeat the proof for the univariate case.

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